A mixed approach to finite element analysis of hyperbolic heat conduction problems

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1. Introduction

Many new engineering applications in heat conduction problems involve considerably high heating rates. This is especially important when a physical process occurs in a time interval comparable to the relaxation time of the medium, or at cryogenic temperatures which correspond to low thermal propagation speed. In such cases, the change in temperature distribution occurs so fast that there is not enough time to reach the thermodynamic equilibrium state. In other words, heat does not transfer with an infinite speed as predicted by the so-called Fourier law. Instead the temperature distribution in the body exhibits a thermal wave travelling with a certain speed. This speed is directly related to the relaxation time τ which is a thermodynamic property of the material and must be determined experimentally. The relaxation time τ represents the phase-lag in time which means that it takes a time τ for a temperature gradient to produce a heat flux. The relaxation parameter τ ranges from 10^{-10} seconds for gases to 10^{-14} seconds for metals.

In general, this non-Fourier effect is shown to decay quickly, but Yuen and Lee (1989) showed that the effect can be important even at a long time after the initial transient if the thermal disturbance is oscillatory with the period of oscillation of the same order of magnitude as the thermal relaxation time. Barletta and Zanchini (1996) have recently studied the thermal resonance phenomenon which remains in a steady state regime.

In order to accommodate this behavior in the heat conduction equation, a modified (non-Fourier) constitutive equation for heat flow has to be invoked. In 1948, Cattaneo proposed a generalized law, the relaxation model, by introducing the time derivative of the heat flux in the Fourier hypothesis (Cattaneo, 1948). The relaxation model has been widely used in many recent publications. Tzou (1992) has recently generalized the relaxation model further by introducing the phase-lag concept.

There are a large number of published papers which can be found in the literature. State-of-the art reviews by Joseph and Preziosi (1989; 1990) and Özisik and Tzou (1994) provide an excellent list of references for an interested reader. Most of the studies on non-Fourier heat conduction (NFHC) have focused

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on simple boundary value problems which can be solved in closed form (e.g. Özisik and Vick, 1984). There are also a few numerical simulations of the NFHC reported for problems in a one-dimensional domain (e.g. Glass *et al.*, 1985; Sadeghipour and Manzari, 1990). Sadeghipour and Manzari (1990) have proposed a finite element method for solving the NFHC equations. In that work, the final governing equations were written in terms of temperature only. The first author of this paper also developed a simple method to predict the temperature jump due to thermal shocks (Manzari, 1990).

Although most published works focus on simple boundary value problems, practical application of the NFHC theory requires the capability to treat very complex geometries as well as more complicated boundary conditions. Numerical methods such as the finite element method are excellent alternatives for development of a general framework which can treat complex geometries and boundary conditions.

In this work, the major goal is to develop a finite element method which treats the heat flux as an independent variable in addition to temperature. Such inclusion of the heat flux as an independent unknown in the final matrix equations greatly enhances the flexibility to impose any boundary condition and to calculate both temperature and the heat flux with the same degree of accuracy.

2. Governing equations

Conservation of energy for an infinitesimal element of a solid body requires

$$q_{i,i} + \rho C_p \dot{T} = g \tag{1}$$

where q_i is the heat flux in the ith direction of a Cartesian co-ordinate system, T is temperature. The notation i denotes a partial derivative with respect to i and a superposed dot stands for a time derivative. In this equation, ρ density and C_p specific heat at constant pressure are material properties and g is the heat generation function. Coleman *et al.* (1982) have shown that the first law of thermodynamics has to be modified for NFHC problems but Bai and Lavine (1995) have shown that for many engineering applications there was no need for such modifications.

A commonly used non-Fourier heat conduction equation can be written as

$$\mathbf{q} + \boldsymbol{\tau} \cdot \dot{\mathbf{q}} = -\mathbf{K} \cdot \nabla T \tag{2}$$

where $\mathbf{K}$ and τ are tensors of thermal conductivity and relaxation parameter of the medium respectively, defined as

$$\mathbf{q} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} K_x & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & K_z \end{bmatrix} \qquad \boldsymbol{\tau} = \begin{bmatrix} \tau_x & 0 & 0 \\ 0 & \tau_y & 0 \\ 0 & 0 & \tau_z \end{bmatrix} \tag{3}$$

Here τ_i is the relaxation time parameter which is defined as α_i/c_i^2, where α is the thermal diffusivity and c_i is the speed of thermal wave in the medium. The case $\tau = 0$ corresponds to Fourier heat conduction equation.

Initial condition

The following initial conditions are usually applied

$$T = T_0 \qquad \dot{T} = 0 \tag{4}$$

where T_0 is the initial temperature of the domain.

Boundary condition

Various boundary conditions may be considered in a heat conduction problem. Details of different boundary conditions are discussed in the following.

The temperature boundary condition states

$$T = T_s \quad \text{on} \quad S_T \tag{5}$$

where S_T is the portion of boundary on which a specified temperature is imposed.

To specify the surface heat flux, q_s, the following boundary condition must be used

$$q_i n_i = -q_s \tag{6}$$

on S_q, where n_i is the ith component of the unit vector outward normal to the boundary. Note that the negative sign denotes inflow heat flux to the domain.

If convective heat transfer is taking place on some part of the body, then the following condition has to be imposed

$$q_i n_i = h(T_s - T_e) \tag{7}$$

on S_c, where h is the convective heat transfer coefficient and T_e is the ambient temperature and T_s is the surface temperature of the body. If radiation heat transfer is also encountered, the following equation can be used

$$q_i n_i = h_r(T_s - T_r) \tag{8}$$

on S_r , where T_r and T_s are temperatures of the radiative source and surface temperature of the body respectively, and h_r is defined as follows

$$h_r = k_r(T_r + T_s)(T_r^2 + T_s^2) \tag{9}$$

where k_r includes the Stefan-Boltzmann constant, the emissivity and the shape factor.

In general, various combinations of the boundary conditions may exist.

3. Finite element formulation

The Galerkin weighted residual method is used to transform the governing differential equations (1) and (2) to integral form. The integral forms are then discretized by applying a standard finite element technique Zienkiewicz and Taylor, 1988).

Applying the Galerkin weighted residual method to equation (1), we have

$$\int_{\Omega} w_1(\rho C_p \dot{T} + q_{i,i} - g)d\Omega = 0 \tag{10}$$

$$\int_{\Omega} \mathbf{w}_2(\mathbf{q} + \boldsymbol{\tau} \cdot \dot{\mathbf{q}} + \mathbf{K} \cdot \nabla T)d\Omega = 0 \tag{11}$$

where w_1, w_2 are weighting functions. Applying the Green's theorem to equation (10) leads to

$$\int_{\Omega}(\rho C_p w_1 \dot{T} - w_{1,i} q_i - g w_1)d\Omega + \int_S q_i n_i w_1 dS = 0 \tag{12}$$

Equation (11) is already in suitable form for finite element solution.

The finite element technique is now utilized to discretize the problem to a number of mixed finite elements. The temperature and heat fluxes' distributions within each element may be approximated using different interpolation functions as

$$\begin{aligned} T(x,y,t) &= \sum_{j=1}^{l} T_j(t)\ {}^T N_j(x,y) \\ q_x(x,y,t) &= \sum_{j=1}^{m} q_{x_j}(t)\ {}^q N_j(x,y) \\ q_y(x,y,t) &= \sum_{j=1}^{m} q_{y_j}(t)\ {}^q N_j(x,y) \end{aligned} \tag{13}$$

where ${}^T N_j$ and ${}^q N_j$ denote the interpolation function at node j of a generic element for the temperature and the flux distribution respectively. The l and m are the number of nodes in the element used to interpolate the temperature and the flux distribution respectively. To accommodate the different interpolation functions used to interpolate the temperature and the flux distribution within each finite element, a mixed ($T6 - q3$) triangular element is used as shown in Figure 1.

Introducing the approximation series (13) into equations (11) and (12) leads to

$$[\mathbf{M}]\{\dot{\boldsymbol{\Delta}}\} + ([\mathbf{K}] + [\mathbf{K}_s])\{\boldsymbol{\Delta}\} = \{\mathbf{F}\} + \{\mathbf{F}_s\} \tag{14}$$

where typical components (e.g. i-j component) of the above matrices and vectors for a 2D domain are defined as

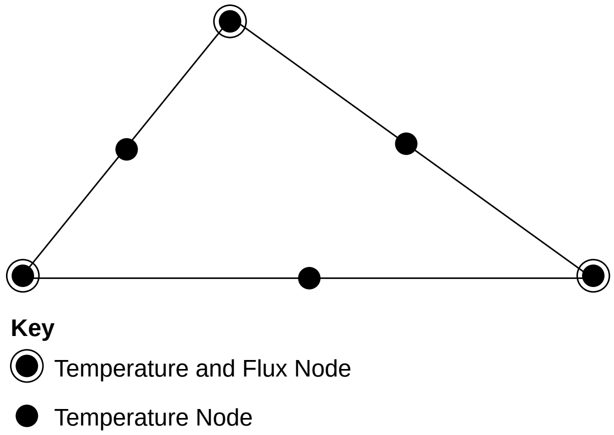

Figure 1. A typical element and its nodal variables

$$M_{ij} = \begin{bmatrix} \int_\Omega \rho C_p K_x K_y{}^T N_i{}^T N_j dxdy & 0 & 0 \\ 0 & \int_\Omega \tau_x K_y{}^q N_i{}^q N_j dxdy & 0 \\ 0 & 0 & \int_\Omega \tau_y K_x{}^q N_i{}^q N_j dxdy \end{bmatrix} \quad (15)$$

$$K_{ij} = \begin{bmatrix} 0 & -\int_\Omega K_x K_y \frac{\partial^T N_i}{\partial x}{}^q N_j dxdy & -\int_\Omega K_x K_y \frac{\partial^T N_i}{\partial y}{}^q N_j dxdy \\ \int_\Omega K_x K_y \frac{\partial^T N_j}{\partial x}{}^q N_i dxdy & \int_\Omega K_y{}^q N_i{}^q N_j dxdy & 0 \\ \int_\Omega K_x K_y \frac{\partial^T N_j}{\partial y}{}^q N_i dxdy & 0 & \int_\Omega K_x{}^q N_i{}^q N_j dxdy \end{bmatrix} \quad (16)$$

$$F_i = \begin{bmatrix} \int_\Omega K_x K_y{}^T N_i g dxdy \\ 0 \\ 0 \end{bmatrix} \quad (17)$$

$$K_{sij} = \begin{bmatrix} \int_S h K_x K_y{}^T N_i{}^T N_j dS & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (18)$$

$$F_{si} = \begin{bmatrix} \int_S (hT_e + q_s) K_x K_y{}^T N_i dS \\ 0 \\ 0 \end{bmatrix} \quad (19)$$

$$\Delta_i = \begin{bmatrix} T_i \\ q_{x_i} \\ q_{y_i} \end{bmatrix} \quad (20)$$

in which i and j are two arbitrary nodes of the element. In equations (18) and (19) it is assumed that we have both the specified heat flux q_s and the convective heat transfer, characterized by h and T_e, on the boundary of the element as

$$q_i n_i = h(T_s - T_e) - q_s \tag{21}$$

The radiative boundary condition can be handled in the same way as the convective boundary condition by replacing h by h_r and T_e by T_r.

Time integration scheme
The system of the ordinary differential equation (14) has to be integrated in terms of time to determine the nodal unknowns at a certain time, e.g. $t + \Delta t$. Equation (14) can be written in the general form of

$$[\mathbf{M}]\{\dot{\Delta}\} + [\bar{\mathbf{K}}]\{\Delta\} = \{\bar{\mathbf{F}}\} \tag{22}$$

where $[\bar{\mathbf{K}}] = [\mathbf{K}] + [\mathbf{K}_s]$ and $\{\bar{\mathbf{F}}\} = \{\mathbf{F}\} = \{\mathbf{F}_s\}$. Using a one parameter (β) interpolation scheme with ($0 < \beta < 1$) , we have

$$\beta\{\dot{\Delta}\}_{t+\Delta t} + (1-\beta)\{\dot{\Delta}\}_t = \frac{\{\Delta\}_{t+\Delta t} - \{\Delta\}_t}{\Delta t} \tag{23}$$

Then equation (22) can be written at time $t + \Delta t$ as follows

$$[\mathbf{M}]\{\Delta\}_{t+\Delta t} = [\mathbf{M}]\{\Delta\}_t + \beta\Delta t(\{\bar{\mathbf{F}}\}_{t+\Delta t} - [\bar{\mathbf{K}}]\{\Delta\}_{t+\Delta t}) + (1-\beta)\Delta t(\{\bar{\mathbf{F}}\}_t - [\bar{\mathbf{K}}]\{\Delta\}_t) \tag{24}$$

or

$$\left(\frac{[\mathbf{M}]}{\Delta t} + \beta[\bar{\mathbf{K}}]\right)\{\Delta\}_{t+\Delta t} + \left(-\frac{[\mathbf{M}]}{\Delta t} + (1-\beta)[\bar{\mathbf{K}}]\right)\{\Delta\}_t = (1-\beta)\{\bar{\mathbf{F}}\}_t + \beta\{\bar{\mathbf{F}}\}_{t+\Delta t} \tag{25}$$

where $[\mathbf{M}]$ and $[\bar{\mathbf{K}}]$ are computed at time t. Therefore, equation (25) can be solved to determine $\{\Delta\}$ at time $t + \Delta t$ explicitly.

The integration method is unconditionally stable for $\beta = 0.5$ (Crank-Nicolson scheme) and $\beta = 2/3$ (Galerkin method).

4. Test problems

In order to verify the validity and efficiency of the proposed method, two one-dimensional test problems for which exact analytical and/or numerical solutions exist are considered first. This is followed by solving two sample 2D problems to demonstrate the capability of the proposed method in dealing with multidimensional problems.

Semi-infinite medium subjected to a step change in the surface heat flux
Consider non-Fourier heat conduction in a semi-infinite medium, initially maintained at zero temperature. At time $t = 0$, the boundary surface at $x = 0$ is exposed to a constant heat flux q_s. The same surface radiates to the surrounding which maintains at $T_r = 0$. The governing equations in dimensionless form become (Glass *et al.*, 1985)

$$\begin{aligned} \frac{\partial Q}{\partial \theta} + \frac{\partial u}{\partial \eta} &= -2Q \\ \frac{\partial Q}{\partial \eta} + \frac{\partial u}{\partial \theta} &= 0 \end{aligned} \tag{26}$$

where the non-dimensionalized temperature

$$u = \frac{T - T_0}{T_R} \qquad Q = \frac{q}{q_s} \tag{27}$$

are functions of η and θ defined as

$$\theta = \frac{t}{2\tau} \qquad \eta = \frac{x}{2\sqrt{\alpha\tau}} \tag{28}$$

where T_R is a reference temperature defined as

$$T_R = \frac{q_s}{k}\sqrt{\alpha\tau} \tag{29}$$

and q_s is the specified heat flux on the boundary. The initial conditions are

$$u(\eta, 0) = 0 \qquad Q(\eta, 0) = 0 \tag{30}$$

and the boundary condition is written as

$$Q(0, \theta) = K_R \left[u^4(0, \theta) - u_r^4\right] - 1 \tag{31}$$

with

$$K_R = k_r q_s^3 \alpha^2 \tau^2 / k^4 \tag{32}$$

The radiation parameter, K_R, depicts the effect of surface radiation on the temperature distribution. Figure 2 shows the temperature distribution in the medium at $\theta = 0.5$ and $\theta = 1.0$ respectively, for $K_R = 0.0, 0.1$. Figure 3 shows the variation of the surface temperature with time. It is seen that due to heat radiation to the ambient, the surface temperature for $K_R = 0.1$ is less than the surface temperature for no radiation case. In these figures the analytical solution obtained by Maurer and Thompson (1973) for $K_R = 0$, and the finite difference solution of Glass *et al.* (1985) and the integral equation solution of Wu (1988), which coincide with each other, for $K_R = 0.1$ are also shown. As it can be seen, the results of the proposed numerical scheme are in good agreement with the analytical solutions and capture the surface temperature jump and the location of the wave front accurately. The small oscillation seen in Figures 2 and 3 could be alleviated by using a finer grid and a smaller time step.

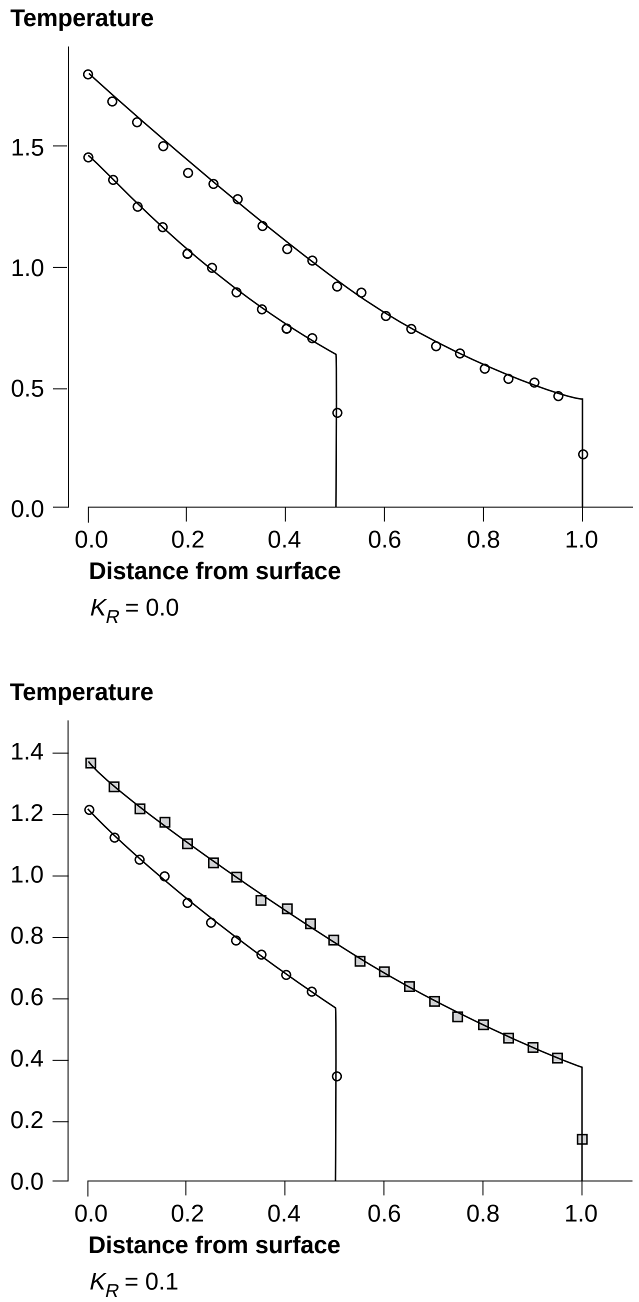

Figure 2.
Step change in surface heat flux; bottom curves (θ = 0.5), top curves (θ = 1.0)– solid lines for K_R = 0.0: (Maurer and Thompson, 1973), solid lines for (K_R = 0.1):(Wu, 1988), circle: FEM

Semi-infinite medium subjected to a step change in the surface temperature
The above problem is solved with different boundary condition. The surface temperature is increased to a constant value T_s at t= 0. Using a different

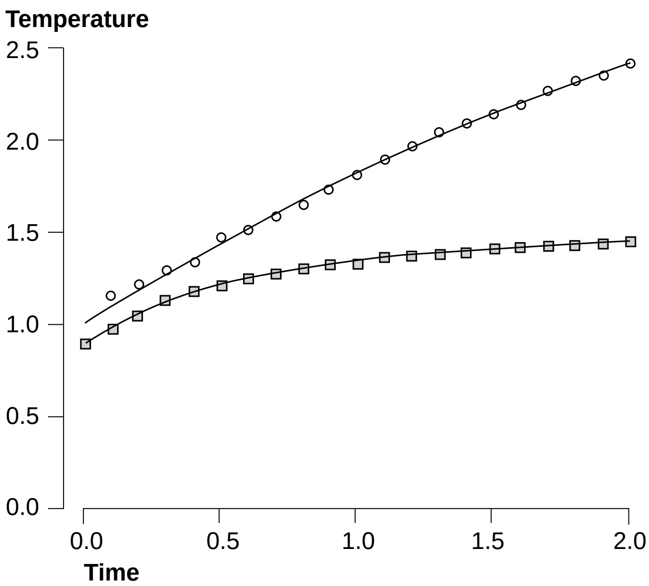

Figure 3. Step change in surface heat flux; surface temperature variation; top curve (K_R = 0.0) – solid line: (Maurer and Thompson, 1973), circle: FEM, bottom curve (K_R = 0.1) – solid line: (Wu, 1988), square: FEM.

definition for the reference temperature as $T_R = T_s - T_0$, the mathematical formulation of this problem becomes similar to the previous case except for the surface boundary condition which is given as $u(0,\theta) = 1.0$. The numerical solution obtained from the proposed finite element formulation is illustrated in Figure 4 at θ =0.5 and θ = 1.0. The analytical solution of Baumeister and Hamill (1969) is also shown in the same figure for comparison. Again good agreement with the analytical solution is achieved and the solution could be improved by using a finer grid and a smaller time step as suggested in the previous test case.

Two-dimensional problems

Two sample problems are considered here to illustrate the application of the proposed method in multidimensional cases. These problems are envisioned to be the two-dimensional counterparts of the above test problems. Figure 5 shows a schematic of the domain and its boundary conditions for two different cases we consider here.

A step change in the surface heat flux

Consider an isotropic square body, initially maintained at zero temperature, the non-Fourier heat conduction equation in dimensionless form becomes

$$\begin{aligned} \frac{\partial Q}{\partial \theta} + \frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi} &= -2Q \\ \frac{\partial Q}{\partial \eta} + \frac{\partial u}{\partial \theta} &= 0 \\ \frac{\partial Q}{\partial \xi} + \frac{\partial u}{\partial \theta} &= 0 \end{aligned} \tag{33}$$

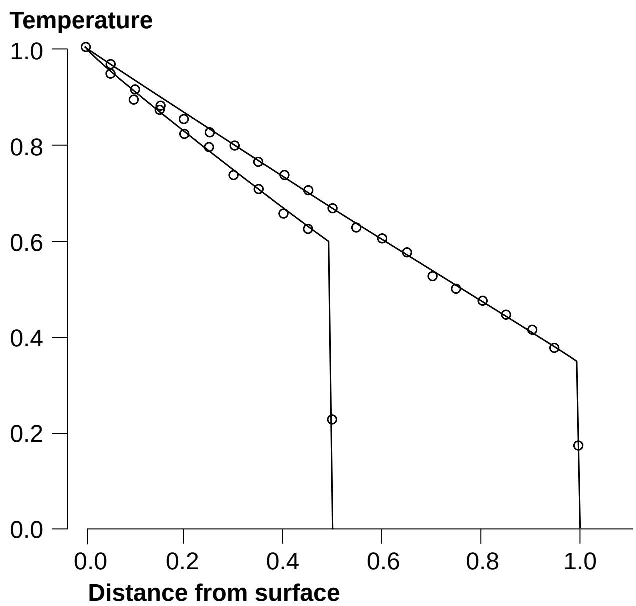

Figure 4.
Step change in surface temperature; temperature distribution inside domain at $\theta = 0.5$ (bottom curve) and $\theta = 1.0$ (top curve). Solid line: Baumeister and Hamill (1969), circle: present method

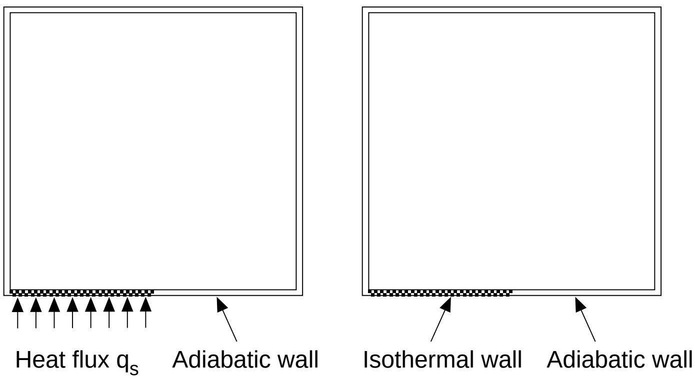

Figure 5.
Domain and boundary conditions for 2D test cases

for $0 \le \eta \le 1$ and $0 \le \xi \le 1$. Suppose that at time $t = 0$, surface temperature of the body in the interval $0 \le \eta \le 0.5$ is exposed to a constant heat flux q_s, while the rest of the boundary is adiabatic. Figure 6 shows isotherms and the q_η and q_ξ contours at $\theta = 0.5$ and $\theta = 1.0$. This figure visualizes moving thermal waves travelling inside the domain. In Figure 7 dotted lines show the evolution of all three primitive variables with respect to time for a point in the middle of the domain. The time delay for wave to reach this point is clearly due to wave nature of phenomenon. The oscillation before and after wave front may be attributed to the time integration scheme employed in this work. It can be shown that a better short-time accuracy is achieved by using the Galerkin

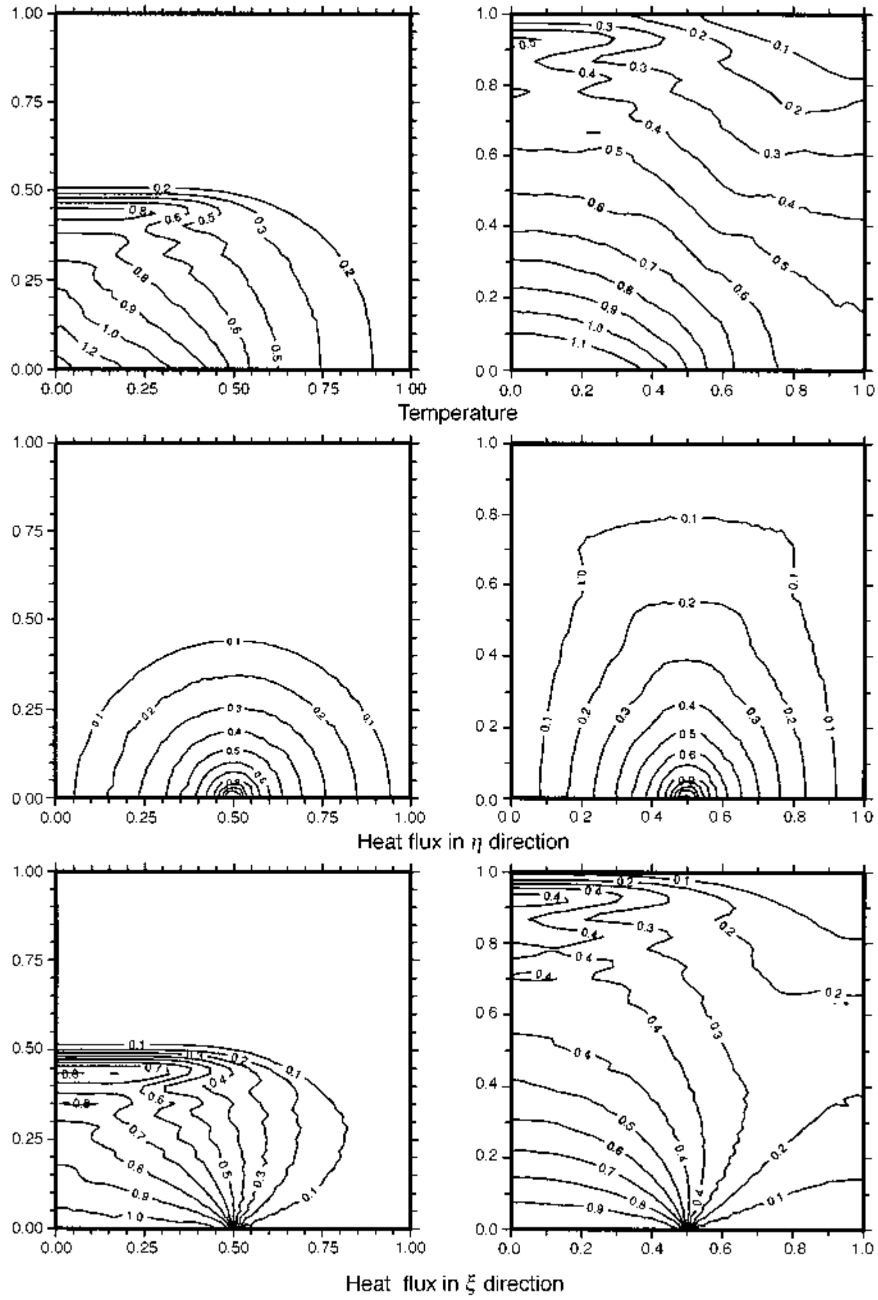

Figure 6.
Step change in surface heat flux; left column: θ = 0.5, right column: θ = 1.0

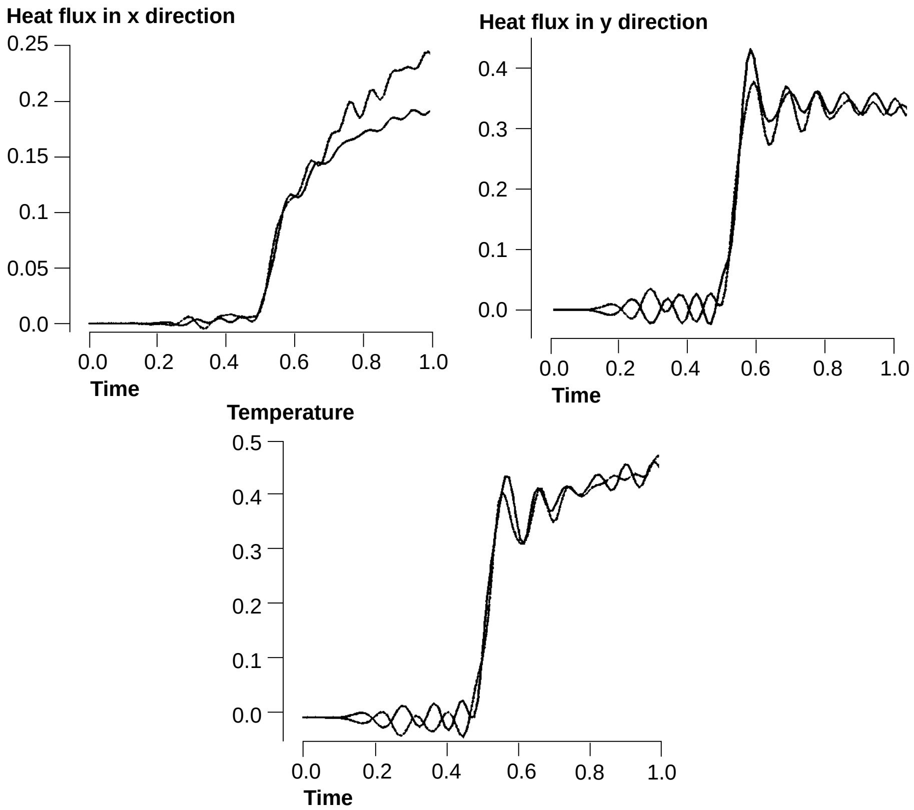

Figure 7.
Temperature and heat fluxes evolution in the middle point; dotted lines: step change in heat flux, solid lines: step change in surface temperature

method (β = 2/3) rather than using the Crank-Nicholson (β = 1/2) method (Lewis *et al.*, 1996).

A step change in the surface temperature
Consider the previous problem with different boundary condition. Suppose that the surface temperature in the interval $0 \leq \eta \leq 0.5$ suddenly rises to a constant value, T_s, at time $\theta = 0$. Figure 8 shows the computed isotherms along with the q_η and q_ξ contour plots at $\theta = 0.5$ and $\theta = 1.0$. An almost similar phenomenon is observed here as in the previous case. Again the evolution of temperature and heat fluxes in the middle point of the domain are shown by solid lines in Figure 7.

5. Conclusions

A mixed finite element approach was devised to solve the non-Fourier heat conduction problems. Owing to flexibility in imposing different boundary conditions and because of simultaneous solution for temperature and heat flux components, the proposed method appears to be effective in dealing with problems of heat wave propagation. The solution of 2D test cases confirms that the method can correctly capture physics of the heat wave motion.

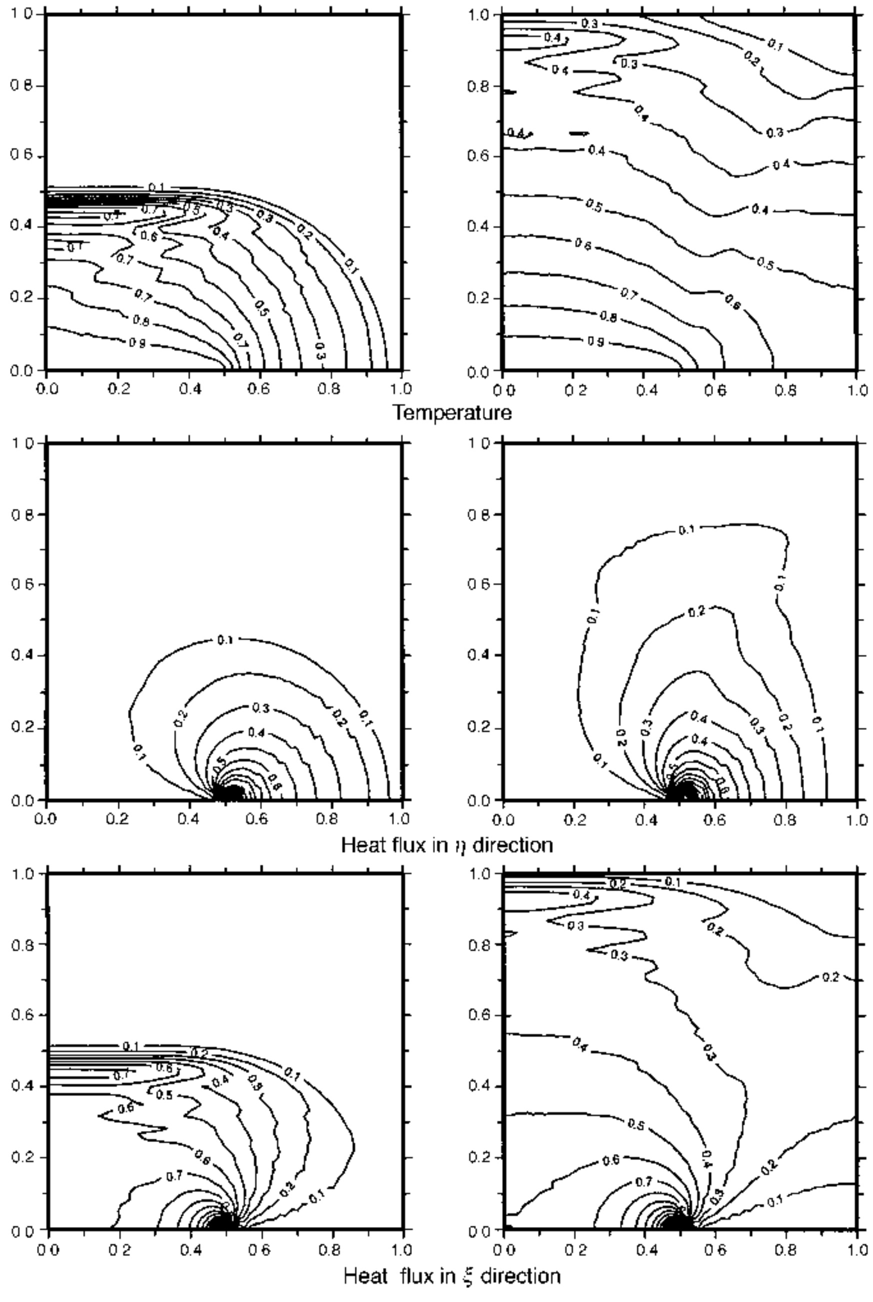

Figure 8.
Step change in surface temperature; left column: $\theta = 0.5$, right column $\theta = 1.0$

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